Microscopic modeling of multi-lane highway traffic flow

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We discuss a microscopic model for the study of multi-lane highway traffic flow dynamics. Each car experiences a force resulting from a combination of the desire of the driver to attain a certain velocity, aerodynamic drag, and change of the force due to car–car interactions. The model also includes multi-lane simulation capability and the ability to add and remove obstructions. We implement the model via a Java applet, which is used to simulate traffic jam formation, the effect of bottlenecks on traffic flow, and the existence of light, medium, and heavy traffic flow. The simulations also provide insight into how the properties of individual cars result in macroscopic behavior. Because the investigation of emergent characteristics is so common in physics, the study of traffic in this manner sheds new light on how the micro-to-macro transition works in general. © *2003 American Association of Physics Teachers.*

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I. INTRODUCTION

We present a microscopic model for traffic flow that describes individual cars and how they interact. Traffic motion is modeled within the framework of Newtonian dynamics. Each car experiences a force resulting from a combination of the desire of the driver to attain a certain velocity, aerodynamic drag, and change of the force due to car–car interactions. The model also includes multi-lane simulation capability and the ability to add and remove obstructions. The model captures the three observed regimes of traffic flow: light traffic, in which cars are well-separated; medium traffic, which includes interacting cars with heterogeneous traffic density along the road; and heavy traffic, in which each car's motion is determined by the spacing allowed by the high concentration. By complementing the simulation with simple analytical results, the model allows students to grasp the main features of traffic flow and how they emerge from the underlying microscopic relations. The model is consistent with experimental observations. We include some open questions that students can study by doing simulations. We hope that it will provide the impetus for further analytical modeling as well.

Most of us have had direct and personal experience with traffic flow. In this paper, we show how a microscopic view of traffic flow, based on an analysis of individual cars and their interactions, relates to our experience. In implementing this model, we provide another perspective to an enduring and fundamental question: how do macroscopic phenomena emerge from individual parameters, governing laws, and interactions?

The study of traffic flow has attracted considerable interest in recent years because of its intrinsic importance.[1–7] Two recent articles provide a comprehensive review of models for traffic flow and related systems.[8,9] The Traffic Forum is a useful repository of recent developments in traffic science,[10] and also provides links to some existing applets.[11]

In traffic flow the interactions of the cars have some special features.[8] For example, the motion is ''self-driven,'' which can be modeled by invoking a driving force. The interaction between cars, for example braking to avoid a car, is modeled as an inter-car force. Newton's third law does not apply to this force, because the influence of car i on car j is not the same as the influence of car j on car i. For example, if car i brakes to avoid hitting car j, car j does not experience the same braking force. The forces propelling each car are real, in the sense that they dictate the dynamics of each vehicle. However, the value of the force is determined by a driver in response to the desired velocity and neighboring vehicles.[8]

There are broadly two approaches toward the development of a model for traffic flow. The first is macroscopic and is based on a classical continuum/statistical mechanics approach, which leads to governing equations based on macroscopic variables. For example, applying such an approach to fluid flow leads to the Navier–Stokes equations.[12] Similar theories exist for traffic flow.[13,1] This approach is the oldest and most direct, but it can be opaque to a beginning student and requires considerable mathematical sophistication.

Furthermore, such treatments invariably simplify the defining dynamics in order to facilitate their execution. As a result, these theories often do not capture the complexities of the system. Nevertheless, such a framework is very important because it can identify key aspects of a system.

The second approach involves discrete simulation using molecular dynamics,[14] Monte Carlo, or cellular automata.[15] These microscopic approaches involve treating each car individually. One considers the state of every car and alters each state according to some algorithm. However, generating the complex behavior on a macroscopic scale often requires vast amounts of data and computations, making the ultimate restriction on a simulation the time needed to complete the task.

In this paper we rely on microscopic simulations augmented by analytical results to establish the framework of traffic flow theory and to introduce its important results. In addition to introducing the specifics of traffic flow statistics, we anticipate that the procedure will instill in the student an appreciation for how a microscopic theory is developed.

The paper is organized as follows. In Sec. II we have collected the definitions of the parameters required to describe traffic flow. Section III describes the basic elements of the model: the governing equations, single-car behavior, car–car interactions, and lane-changing rules. The simulation, TrafficMaker,[16] has been implemented as an applet. Section III contains a series of suggested simulations using

Table I. Summary of the quantities of the traffic flow model. M indicates that the mathematical definition of that quantity is model dependent. The relationships are $q = c\overline{v}$ and $s_i = h_i v_i$.

Quantity	Definition (in words)	Definition (math)	Units
x_i	position of *i*th car	N/A	m
v_i	velocity of *i*th car	N/A	m/s
m_i	mass of *i*th car	N/A	kh
l_i	length of *i*th car (+ minimum clearance)	N/A	m
L	Road length	N/A	m
N_ℓ	number of cars in Δx of lane ℓ	N/A	cars
q_ℓ	traffic flow past a region Δx of lane ℓ	$\sum_i^N v_i/\Delta x$	cars/s
c_ℓ	density of cars in region Δx of lane ℓ	$N/\Delta x$	cars/m
$\overline{v}_\ell$	mean velocity of cars in region Δx of lane ℓ	$\frac{1}{N}\sum_i^N v_i$	m/s
$s_{i\rightarrow j}$	distance between car i and car j	$x_j - x_i$	m
$h_{i\rightarrow j}$	time needed for car i to drive $s_{i\rightarrow j}$ (headway)	$s_{i\rightarrow j}/v_i$	s
$\Delta v_{i\rightarrow j}$	velocity of car i relative to car j	$v_j - v_i$	m
s_i^*	following distance desired by the driver of car i	M	m
h_i^*	desired headway for car i	M	s
v_i^*	desired velocity for car i	M	m/s
t_i^*	reaction time for car i	M	s
F_i	force caused by driver on car i	M	kg m/s^2
$F_i^{\max}$	maximum accelerating force of car i	M	kg m/s^2
$F_i^{\min}$	maximum breaking force of car i	M	kg m/s^2
$v_i^{\max}$	speed limit for car i	M	m/s

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the program, and introduces single-car behavior, two-car behavior and car–car interactions, macroscopic traffic statistics in a single lane, the formation and dynamics of traffic jams, multiple-lane situations, and jams and traffic flow in multiple lanes.

II. DEFINITIONS

Before presenting the microscopic model in detail, we will discuss some of the fundamental concepts in traffic science.[17] We also provide an extensive list of terms and concepts in Table I. Some notable texts on traffic science suitable for physicists include Refs. 18 and 19.

There are three significant macroscopic variables that describe traffic conditions. First, the flow, q_ℓ for lane ℓ, is the number of cars that pass a point per unit time. It is defined as

$$q_\ell = \frac{1}{L}\sum_i v_i , \tag{1}$$

where L is the length of the highway and the summation is over all the cars in a particular lane. The density, c_ℓ, is given simply by

$$c_\ell = \frac{N_\ell}{L}. \tag{2}$$

Because $\overline{v_\ell} = \Sigma_i v_i / N_\ell$ is the mean velocity, we have the basic relation

$$q_\ell = c_\ell \overline{v}_\ell . \tag{3}$$

Therefore, high flow can result from a few fast moving cars or many slow moving cars.

Microscopically, two important parameters that determine the interaction between cars i and j are their spatial separation, $s_{i\rightarrow j}$, and temporal separation or headway, $h_{i\rightarrow j}$. The two are related by the velocity as

$$s_{i\rightarrow j} = h_{i\rightarrow j} v_i . \tag{4}$$

The notation $\Delta v_{i\rightarrow j} = v_j - v_i$ is used to represent the relative velocity of i and j.

The values of the spacing and headway are important relative to the minimum following distance, s^*, or to the related minimum headway, h^*. Car–car interactions become important when the separation between cars is on the order of s^*. The quantities s^* and h^* are not constants, but depend on the type of each car and its velocity. For simplicity, we take h^* to be constant. That is, we assume each driver aims to maintain a minimum headway to the car ahead, no matter the conditions. Then s^* is given by

$$s_i^* = l_i + h^* v_i . \tag{5}$$

In the program, we have fixed $h^* = 1.25$ s, which is chosen to match approximately the observed maximum highway vehicular flow.[20]

Highway traffic can be grouped into three density regimes: light, moderate, and heavy.[21] Light traffic is characterized by all the cars on the road being able to drive as fast as they desire without having to interact with other cars. Light traffic is similar to a gaseous state. As the number of cars on the road increases, drivers are forced to adjust their driving patterns to accommodate other vehicles. This is the moderate traffic density state. It may be compared to a gaseous state where intermolecular interactions matter. In moderate density traffic flow the homogeneous state tends to be unstable.

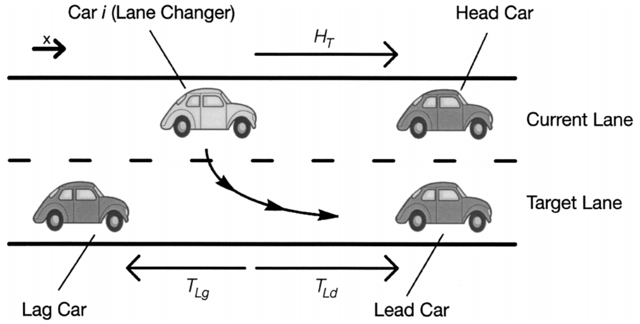

Fig. 1. Illustration of a segment of multilane highway traffic.

Small platoons of cars form due to obstructions by slower cars, but these obstructions are short lived. Over all, the average speed of moderate density traffic is lower than light traffic, but not appreciably. The scattered platooning of cars makes moderate traffic behave similarly to water vapor with slight condensation. Heavy traffic is analogous to the liquid phase. The traffic density is so high in heavy traffic that the entire roadway is congested. Follow-the-leader traffic ensues, and average velocities decline significantly. On a multi-lane road, every lane moves slowly.

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III. ELEMENTS OF THE MICROSCOPIC MODEL

Consider Fig. 1, which shows a schematic drawing of several cars on a highway. Our model consists of a highway of length L with one to three lanes, and a number of cars, N. The current state of the system is described by the location x_i and velocity v_i of each car, indexed by $0 \leqslant i < N$. In Sec. IV, we will develop a microscopic model for the motion of these cars. For this purpose, each car has associated with it a mass, m_i, and length, l_i.

We assume that traffic flow can be cast onto the framework of Newtonian dynamics in the following way. Upon entering the road, each car seeks to accelerate to a desired cruising velocity. This desired velocity is achieved by the car exerting a force against the road, while being resisted by viscous air drag.[22] Isolated cars simply accelerate up to their desired velocity v_i^*. When the density of cars increases, the force applied by each car is decreased in an attempt to maintain a desired following distance. Finally, in a multi-lane setting, cars are allowed to change lanes if frustrated by the presence of a slower car ahead.

A. Dynamics of noninteracting vehicles

Flow charts of the algorithm are given in Figs. 2 and 3. The dynamics of car i is governed by the equation of motion (see Fig. 1)

$$m_i \frac{d^2 x_i}{dt^2} = F_i + F_d(v_i). \tag{6}$$

Equation (6) states that the total force on car i is the force due to the engine (or brakes), F_i, along with the drag force of wind resistance, F_d. F_i is the force applied by the driver of car i. A positive F_i represents the driver using the gas pedal, and a negative F_i represents the driver using the brakes.

At low velocities the air flow is laminar, and the damping force F_d is linear in velocity v_i. At higher velocities, the air flow around the car is turbulent, and F_d is quadratic in v_i. We will take F_d to be a linear function of velocity. Then the governing equation of motion is

$$m_i \frac{d^2 x_i}{dt^2} = F_i - \eta_i \frac{dx_i}{dt}. \tag{7}$$

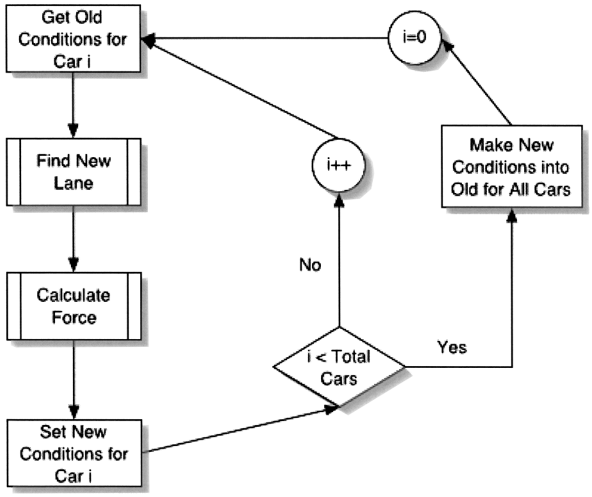

Fig. 2. Flow chart giving an overview of the algorithm.

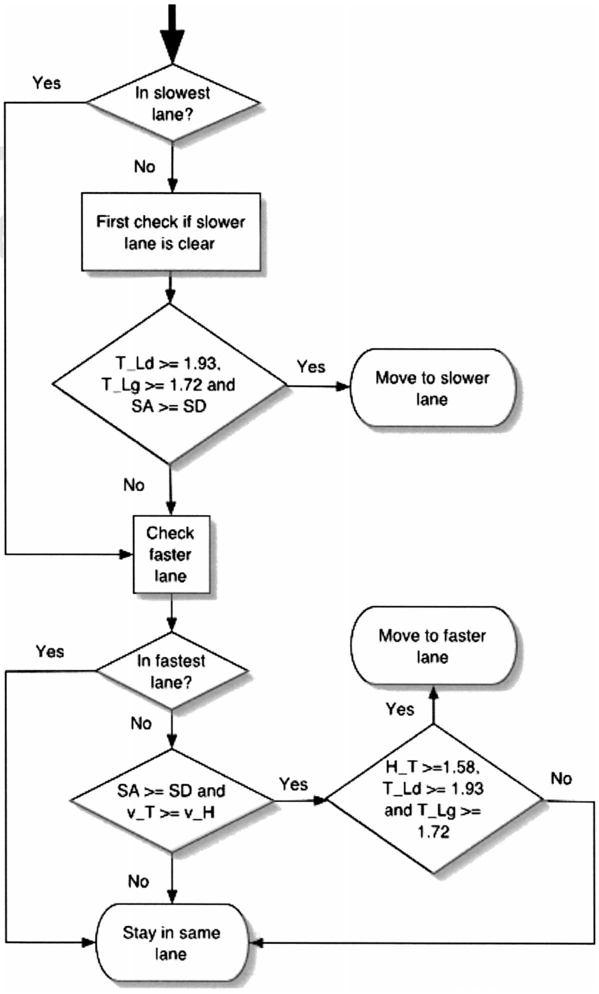

Fig. 3. Flow chart showing the process for changing lanes.

The constant η_i represents an effective frictional viscosity. The constant m_i/η_i represents a characteristic time as discussed in the Appendix. This time represents how long it takes a car to change velocity when the force changes suddenly. If the drag force is assumed to be quadratic in velocity, we would also find that the transient response of a vehicle for a change in force has a characteristic time. Under realistic conditions, the drag force is closer to being quadratic than linear in velocity. However, as shown in the Appendix, once we fix a characteristic time, the linear and quadratic drag terms result in very similar responses. Therefore, we can incorporate the analytical simplicity resulting from the linear drag term without compromising our model.

The general solution for the homogeneous case of Eq. (7) $(F_i=0)$,

$$m_i \frac{d^2 x_i^g}{dt^2} + \eta_i \frac{dx_i^g}{dt} = m_i \frac{dv_i^g}{dt} + \eta_i v_i^g = 0, \tag{8}$$

is

$$v_i^g(t) = A_i e^{-\eta_i t/m_i}, \tag{9}$$

That is, cars achieve their steady-state velocity with the characteristic time, τ_i,

$$\tau_i = \frac{m_i}{\eta_i}. \tag{10}$$

To find the particular solution to Eq. (7), we assume that F_i is independent of both position and time. The particular solution is

$$v_i^p(t) = \frac{F_i}{\eta}. \tag{11}$$

The complete solution is

$$v_i(t) = v_i^g(t) + v_i^p(t) = A_i e^{-t/\tau_i} + \frac{F_i}{\eta_i}, \tag{12}$$

where the constant A_i is determined by the initial condition for the velocity. One of the fundamental features of our model is that an isolated car will achieve a desired steady-state velocity of F_i/η_i as $t \to \infty$. If it started with an initial velocity v_i^0 at $t=0$, then $A_i = (v_i^0 - F_i/\eta_i)$. In our implementation, we take all the cars be the same type, that is, they all have the same mass, length, and characteristic time. We let the characteristic time be $\tau = 8$ s and $m_i = 1000$ kg. Consequently, $\eta = \eta_i = 125$ kg/s.[23] With these assumptions, the velocity of the car is

$$v_i(t) = \left(v_i^0 - \frac{F_i}{\eta} \right) e^{-t/\tau} + \frac{F_i}{\eta}. \tag{13}$$

Hence, F_i/η represents the desired velocity. As we shall see, this velocity will vary depending on the conditions of the surrounding traffic. The position of car i comes from integrating the velocity over time:

$$x_i(t) = -\tau \left(v_0 - \frac{F_i}{\eta} \right) e^{-t/\tau} + \frac{F_i}{\eta} t + x_0. \tag{14}$$

Given a state of the road described by the position and velocities of all cars, the program moves forward an increment of time, Δt, by forward numerical integration of the equation of motion (Euler method), using the values at the beginning of the step. After each time step, the driver adjusts F_i. This forward integration method has the advantage that only the calculation of the force involves interactions with other cars. Therefore, the integration of the motion is independent for each car. This advantage allows us to simulate large numbers of cars easily. However, it suffers from numerical instabilities if the time step is on the order of or larger than the characteristic time.

B. Specifying F_i: The car–car interaction

From the point of view of the driver, the influence of the surroundings is twofold. The presence of a car ahead modifies the force the driver applies. Moreover, the cars ahead and in adjacent lanes may cause the driver to change lanes. We describe here how the force is modified due to the presence of other cars.

Many approaches are possible as was mentioned in Sec. I.[7] We assume that the driver of car i will modify her/his speed based on the position and speed of the car j just ahead of it. That is, the driver places primary importance on the desired following distance, s_i^*. The driver adjusts the force (accelerator or brake) to bring his/her car to the desired following distance as quickly as possible, without exceeding v_i^*. In particular, F_i is chosen to be

$$F_i = \eta v_j + (F_i^{\max} - \eta v_j)[1 - e^{\Delta v_{i\to j}/v'} e^{(s_i^* - s_{i\to j})/s'}]. \tag{15}$$

Here $F_i^{\max}$ is the force necessary to drive the car at velocity $v_i^{\max}$, while v' and s' are the characteristic velocity and distance, respectively. In our implementation, $v' = v_i^*$ and $s' = l_i$.

To see how we arrive at Eq. (15), consider again Fig. 1. Imagine yourself in car i, looking ahead at car j. Consistent with the assumption that the force is determined by the spacing and relative velocity, move to a frame of reference fixed with respect to car j. When a faster car is behind a slower one, we anticipate that in the steady state, the faster one will reduce its force to maintain a fixed distance, s_i^*. Therefore car i must exert a force that allows it to travel at the velocity of car j. In the steady state,

$$F_i = \eta v_j, \tag{16}$$

when $s_{i\to j} = s_i^*$ and $\Delta v_{i\to j} = 0$. Moreover, when either $-s_{i\to j}$ or $\Delta v_{i\to j}$ approach $-\infty$, we have

$$F_i = F_i^{\max}. \tag{17}$$

Both conditions are satisfied by

$$F_i = \eta v_j + (F_i^{\max} - \eta v_j) G(\Delta v_{i\to j}, (s_i^* - s_{i\to j})), \tag{18}$$

if the function $G(x,y)$ has the properties

$$G(0,0) = 0, \tag{19a}$$

$$G(-\infty, y) = 1, \tag{19b}$$

$$G(x, -\infty) = 1. \tag{19c}$$

The choice,

$$G(v_{i\to j}, (s_i^* - s_{i\to j})) = [1 - e^{\Delta v_{i\to j}/v'} e^{(s_i^* - s_{i\to j})/s'}], \tag{20}$$

gives Eq. (15). The results of this definition of F_i satisfy the following intuitively obvious requirements:

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(1) If car i is far behind car j, $F_i \rightarrow F_i^{\max}$.
(2) If car i is much slower than car j, $F_i \rightarrow F_i^{\max}$.
(3) If car i follows car j at a distance s_i^* and $v_i = v_j$, then the force is just enough to maintain the velocity, $\min[v_j, v_i^*]$.
(4) If car i gets closer than s_i^* to car j, there is a strong braking force.
(5) If car i is much faster than car j, car i will begin braking. If car i is far behind car j, then car i will gradually slow down. If car i is close to car j, then car i will brake quickly.

In the simulation, we add two additional constraints. The new velocity of any car is not allowed to be negative, and the new position is not allowed to be behind the current position. For sufficiently small time steps these conditions are satisfied automatically. Enforcing these conditions in the program ensures realistic behavior.

C. Rules for changing lanes

Unlike the car-following behavior of single-lane driving, the act of changing lanes is a discrete, not continuous, process. Hence, we handle lane-changing separately from car-following.

There are two aspects of a lane-changing model. First, the driver must choose the destination lane. Then, the driver must make a yes or no decision to change based on the surrounding traffic.

To decide how a car will change lanes, we assumed that most highways have fast and slow lanes. In keeping with the convention of American highways, the right-most lane is the ‘‘slow’’ lane. Therefore, we want the cars in our simulation to keep to the right-most lane whenever possible. Given a reason to change lanes, a car will switch to the next lane to the left, if such a lane exists, and return to the lane to the right as soon as it is clear.

When determining if a lane change is safe, a driver must consider not only the car directly ahead, but also the lead and lag vehicles in the target lane (see Fig. 1). Because of the number of cars involved in the decision, a lane-changing model will have a significant number of parameters. We have turned to the work of Ref. 24 in which a lane-changing model based on the observation of cars on urban streets of Kansas City, MO, was developed. In our highway traffic simulation, cars execute what Wei *et al.*[24] call a ‘‘discretionary lane change,’’ which is defined as passing ‘‘a slower-moving vehicle.’’[24] In this model, a driver considers three parameters when deciding if a lane change is acceptable: the headway to the head car (H_T), to the lead car (T_{Ld}), and to the lag car (T_{Lg}) (see Fig. 1). According to Ref. 24, a discretionary lane change will be accepted if

$$H_T \geqslant 1.58 \text{ s}, \tag{21a}$$

$$T_{\mathrm{Ld}} \geqslant 1.93 \text{ s}, \tag{21b}$$

$$T_{\mathrm{Lg}} \geqslant 1.72 \text{ s}. \tag{21c}$$

Although these are the conditions for accepting a lane change, what makes a driver decide to change lanes in the first place? In Ref. 24 two quantities are defined, the speed advantage, SA, and speed disadvantage, SD. They are defined as

$$\mathrm{SA} = \frac{v_{\mathrm{Ld}} - v_H}{v_{\mathrm{Ld}}}, \tag{22a}$$

$$\mathrm{SD} = \frac{v_T - v_H}{v_T}. \tag{22b}$$

When SA exceeds SD, changing lanes will allow the car to travel faster. On the other hand, if changing lanes would result in a slower speed, the driver should not do it, so we prevent lane changes when SD exceeds SA or if the speed disadvantage SD is negative, meaning the head car is traveling faster than the car of interest.

Once the final lane has been determined, the simulation calculates the new velocity of the car based on the car-following model, which means drivers adjust their speed to match the conditions in the new lane, not the old lane.

IV. EXAMPLE SIMULATIONS

We now present a sequence of simulations for students to do using TrafficMaker.[16] These simulations start with simple one and two car simulations on a single-lane road and increase in complexity. We also introduce some principal features and results of traffic flow dynamics.

Figure 4 shows a snapshot image of the program. Traffic moves from left to right. The road can be $\frac{1}{2}$ or 1 mile long, and has periodic boundary conditions at its ends. That is, cars exiting from the right reenter at the left, as if the road were a circular track. The simulation has two types of cars: normal cars and broken-down cars to serve as lane obstructions. Normal cars can be introduced by clicking on the road or added automatically at regular intervals. Broken-down cars can be added and later removed. A check-box enables the visualization of the braking distance ahead of each car. The starting conditions can be light, medium, or heavy traffic. The two panels plot flow statistics.

A. One lane, one car

The simplest simulation involves a single car without any obstructions or outside influences.[25] The car initially starts with a random speed, but notice how it quickly moves to a steady speed. This is the car’s maximum desired speed.

To get a feel for how a car brakes and accelerates, add a broken-down car. A broken-down car will never move.[26] By placing broken-down cars at different distances from the moving car, you can explore how the braking force is a function of distance.

B. One lane, several cars

After exploring single car dynamics, the next step is to add a second car. Start the simulation with a single car on a one-lane road. Then add a second car far away from the first car, so it will reach a steady state without interacting with the first car.[27]

Then extend the simulation to several cars, say 10–20, and observe how they eventually pile-up behind the slowest. Then consider the question: given a distribution of individual car velocities and many realizations of, say, 20 cars, what is the distribution of the mean velocity? Clearly, for a one-lane road, the mean road velocity in the steady state is simply the velocity of the slowest car. This case is an example of extreme-value statistics which is important in several other contexts, for example, in the strength of brittle solids.[28] An

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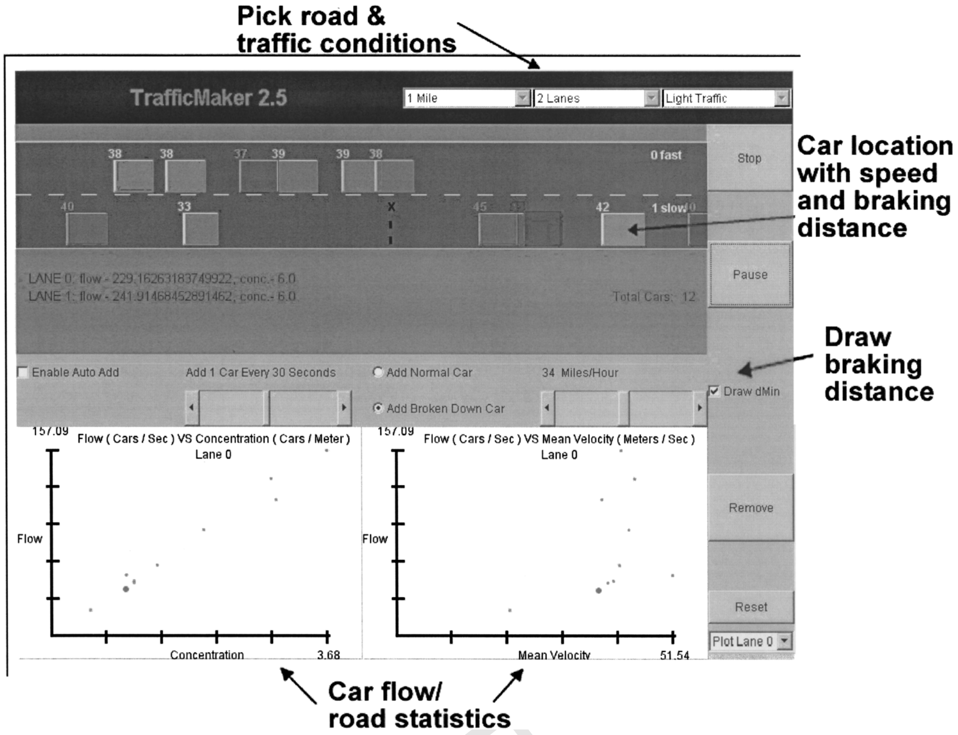

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LANE 0: flow - 229.16263183749922, conc.- 6.0
LANE 1: flow - 241.91468452891462, conc.- 6.0
                                             Total Cars:  12
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Fig. 4. Screen shot of TrafficMaker.

interesting problem is can one derive an analytical expression for the road velocity distribution in this case?

C. One lane, traffic jam

Create a traffic jam by placing a broken-down car in the lane and stacking up cars behind it.

After starting the simulation with a single car as before, place a broken-down car in the center of the road. Add five normal cars and wait for them all to stop behind the broken-down car. Then remove the broken-down car to allow the cars to move again.[29] Notice that the traffic soon returns to normal, with the cars traveling at or close to their maximum desired velocity. Reset the simulation and create another traffic jam, this time increasing the number of cars by two or three. Again, remove the broken-down car and observe the traffic return to normal. Repeat this procedure several times, each time increasing the number of cars in the traffic jam. At a certain number, the traffic will no longer return to normal and the traffic jam will persist. Estimate the velocity of the jam and measure its length.

Consider the following interesting questions:

- A self-created traffic jam moves in the opposite direction from the direction of traffic flow. What is the rate of movement of the traffic jam to the left? To formulate this problem, consider the rate at which cars enter the jam versus the rate at which cars exit the jam.
- What is the condition for a jam to dissipate rather than grow? Simply put, if the flow of cars to the left is greater than the exit flow, the jam will grow, otherwise it will dissipate. If the entering flow is prescribed, the answer of this question requires a solution to the previous problem.

In the bottom half of the applet, two graphs show a running plot of flow versus concentration and flow versus mean

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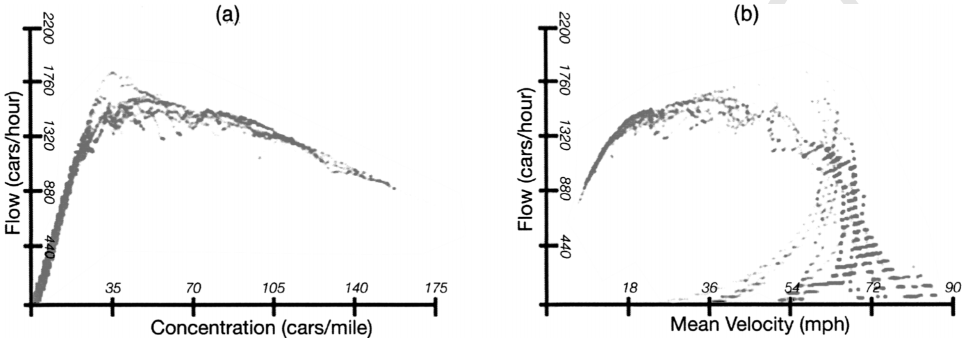

Fig. 5. Flow statistics for single lane highway traffic. In both plots, the darker the dots, the more common the appearance of that particular road condition. (a) As the concentration of cars increases, flow increases linearly until the transition to the medium traffic regime, indicated by the peak around 35 cars/mile. After the medium traffic regime has been reached, additional cars only obstruct each other, reducing the flow. (b) The plot of mean velocity versus flow shows that the highest flow does not necessarily correspond to the highest mean velocity. High flow can result from many cars driving slowly or a few cars driving quickly.

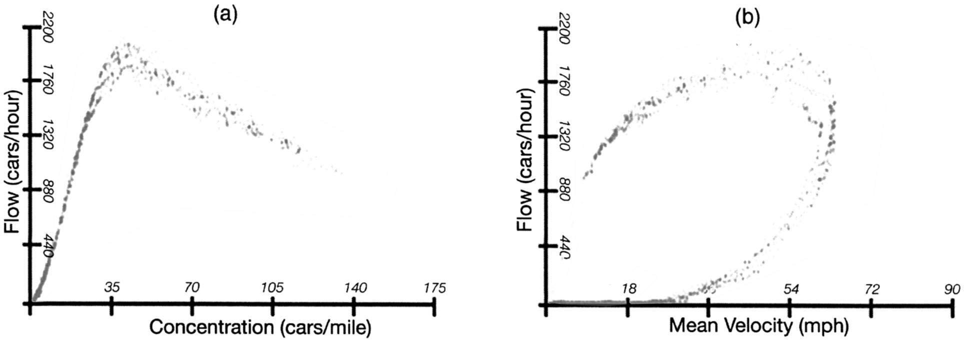

Fig. 6. Flow statistics for two lane highway traffic: Slow lane. (a) Even in the multi-lane case, the flow versus concentration curve retains its characteristic peak. (b) However, because fast cars move to the fast lane, the bottom right corner of the flow versus mean velocity curve is now essentially a single trace instead of the wide base in Fig. 5 due to the homogeneity of velocities.

velocity. This information provides quantitative feedback and is often used to describe traffic conditions.[20,30–32] Allow sufficient concentration to build up to see data like those shown in Figs. 5(a) and 5(b).[33]

Both the flow/concentration and flow/mean velocity curves have three distinct types of behavior, corresponding to light, medium, and heavy traffic flow. In the flow/concentration curve the straight line extending from the origin indicates that cars are far enough apart to not interact, so every new car drives at its desired speed and increases the flow proportionally. Once the cars begin to interact, some cars must slow down. Traffic is not dense enough to force all of the cars to slow down, allowing a broad distribution of velocities. However, the flow and mean velocity constantly change, because sometimes fast cars are forced to brake and slow cars drive at full speed, while other times the situation is reversed. This fluctuation produces irregular oscillations in the middle section of the curve, due to the time-dependent nature of the flow. When the concentration of cars increases to the point that all of the cars must slow down, each additional car forces every other car to brake accordingly and the curve suddenly stabilizes into the linearly decaying segment indicating heavy traffic. Notice that the same flow is achieved at two different concentrations.

As cars are added, they accelerate or decelerate to their desired speed. Because the desired speeds are distributed about 65 mph in our model, in light traffic the mean velocity approaches 65 mph with each new car added. The transition from light to medium traffic is indicated by the divergence of the previously converging traces. Just like the flow/concentration curve, the portion of the flow/mean velocity curve representing medium traffic is irregular. Once the traffic becomes heavy, the graph converges once again. The graph becomes especially narrow due to constant obstruction by the other cars, greatly reducing the range of possible velocities.

We can derive simple analytical expressions for steady state flow in the light and heavy traffic regimes. In the latter regime we assume that every car is traveling at the same velocity and by definition has the same s_i^*. Then the concentration and the distance between the cars is given by

$$s_i^* = \frac{1}{c} = (l + h^* \overline{v}). \tag{23}$$

A simple rearrangement shows that the relation between flow and concentration in the congested (heavy traffic) case is

$$q = \frac{1}{h^*} - \frac{cl}{h^*}. \tag{24}$$

For data and fits from real traffic measurements that are consistent with this model, see Ref. 30. The derivation of Eq. (24) is a simple way of understanding the linear reduction in flow with increasing concentration [see Fig. 5(a)]. For light traffic, each car travels at its peak velocity. Then

$$q = \frac{1}{L} \sum_i v_i^* = \frac{N\langle v^* \rangle}{L} = c\langle v^* \rangle. \tag{25}$$

This regime represents the increasing portion of the plot in Fig. 5(a) at small concentrations.

If we equate the flow-concentration expressions in Eqs. (24) and (25), we find that the transition between light and heavy traffic occurs approximately at a concentration of

$$c_{\mathrm{tr}} = \frac{1}{l + \langle v^* \rangle h^*}. \tag{26}$$

For the parameters of our implementation, $c_{\mathrm{tr}} \sim 37$, consistent with Fig. 5(a). The corresponding maximum flow, or the peak flow capacity of the road is, roughly,

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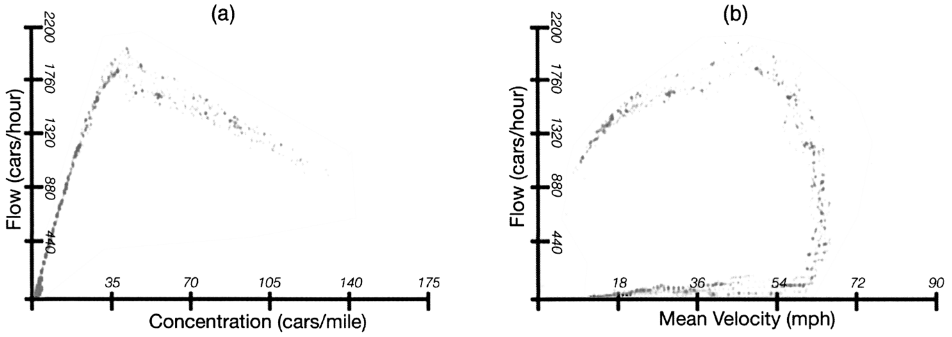

Fig. 7. Flow statistics for two lane highway traffic: Fast lane. (a) Again, the fast lane's flow versus concentration curve also has the characteristic peak, indicating the fundamental nature of the flow versus concentration curve. (b) Under low traffic conditions, the fast lane contains faster cars, hence the higher mean velocity under in the light traffic regime.

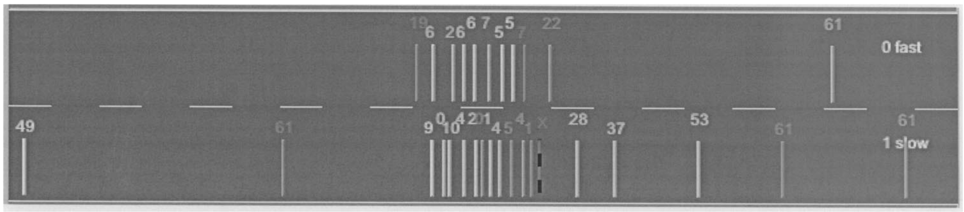

Fig. 8. Screen shot of a two-lane bottleneck.

$$q = \frac{\langle v^* \rangle}{(l + \langle v^* \rangle h^*)}. \tag{27}$$

Statistical variations limit the actual peak flow to be considerably lower. Use the program to judge how accurate these simple analytical results are for the different regimes of traffic flow. The medium flow regime is the most difficult to analyze, because it contains a mixture of congested and freely-flowing regions. Many of the works in traffic flow theory attempt to solve for flow and mean velocity in this regime.[18]

D. Two lanes

Adding a second lane makes a qualitative change; passing transforms the simulation from a relatively simple follow-the-leader process into true highway driving. Because of the added complexities associated with multi-lane driving, it is best to start simply. Start the simulation with two lanes. Notice that regardless of the initial lane, the car immediately moves to the slow lane. Now introduce a broken-down car somewhere in the slow lane. The normal car will switch lanes to go around the obstruction.

E. Two lanes, several cars

Attempt a two-lane simulation with two cars. In the single lane situation, the faster car inevitably became stuck behind the slower car. Now the faster car can pass the slower car, and both can proceed at their desired speeds unhindered. Because the slow car will never pass the fast car, the fast lane will always contain faster cars.

F. Two lanes, many cars

Use the simulation to answer the questions: is the flow in both lanes similar? Does the faster lane actually have a faster mean velocity? Figures 6 and 7 show statistics for the slow and fast lanes. Notice that the flow-concentration curves for the two lanes are remarkably similar, reflecting the underlying microscopic traffic parameters, not individual lane conditions. The differences between lanes are more apparent in the flow-mean velocity diagrams. The fast lane truly is faster only in the light flow regime. In light flow, for the same concentration, the fast lane has faster cars. Once congestion appears, the mean velocity in the two lanes become very similar because the flow is determined by the spacing [see Eq. (24)].

In the one-lane case, a broken-down car completely halts traffic. However, in the multi-lane case a bottleneck forms. Here we investigate how a bottleneck effects traffic flow. Start the simulation with two lanes and put a broken-down car in the slow lane, as shown in Fig. 8. Both lanes merge at the bottleneck, doubling the concentration at that point. If the concentration of both lanes is very low, cars will pass through the bottleneck without slowing down. Once the concentration in the unimpeded lane becomes high enough so that cars in the blocked lane must wait to pull into the clear lane, a queue forms in the blocked lane. Cars in the blocked lane slow down as they wait in the queue, so when they finally pull into the unimpeded lane at their reduced speed, they force the traffic in that lane to slow down as well. Thus, both lanes jam under bottleneck conditions.

After you have observed bottleneck behavior, what sort of flow statistics do you expect? See Figs. 9 and 10 for a case of two-lane highway with a single obstruction in the slow lane. Note the increased scatter in the flow-concentration and flow-mean velocity diagrams. Also, the fast lane enters the medium flow regime at smaller concentration, because near the obstruction the entire flow is forced to go through the fast lane. Locally the concentration is much higher than the mean.

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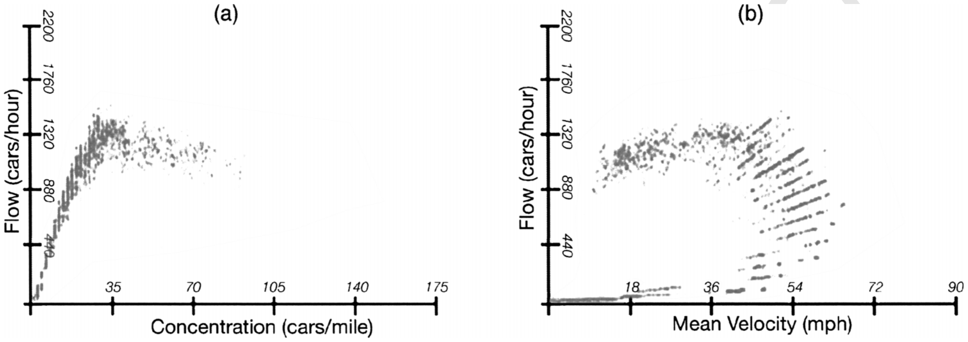

Fig. 9. Flow statistics for two-lane highway traffic with a broken-down car in the slow lane (see Fig. 8). The plot shows the road statistics for the lane containing the blockage. (a) Although we still see the characteristic peak, the curve has a large spread in data values, resulting from conditions around the broken-down car varying over time. (b) The streaks in the mean velocity versus flow curve result after a jam at the bottle-neck clears, allowing the entire roadway to gradually increase speed. When a car changes lanes, the flow changes in an incremental amount, hence even distribution of streaks. Eventually, a jam reforms at the bottle-neck, the cars are forced to slow down, and the cycle begins again.

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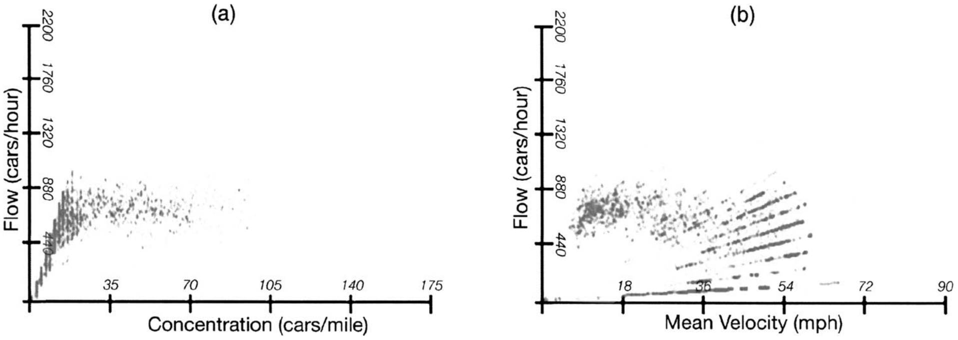

Fig. 10. Flow statistics for two-lane highway traffic with a broken-down car in the slow lane (see Fig. 8). This shows the road statistics for the lane without the blockage, in this case the fast lane. (a) Because cars from both the slow lane and the fast lane must merge to pass through the bottle neck, the clear lane's concentration effectively doubles, hence the peak occurs at half the concentration. (b) Slow cars waiting in the obstructed lane eventually enter into the unblocked lane, resulting in large oscillations in the mean velocity in the unblocked lane.

V. SUMMARY AND CONCLUSIONS

We have presented a microscopic model for traffic flow that shows how macroscopic dynamics emerge from individual car behavior and their interactions. Traffic flow has been modeled from the perspective of the individual driver, making it well suited for simulation. The simulation allows us to capture considerable complexity in car–car interactions and the range of traffic parameters. The traffic flow varies from light, medium, to heavy traffic.

The light and heavy phases of traffic flow are more amenable to analysis. The flow-concentration result for heavy traffic is independent of the specific car–car interaction model. For light traffic, the results depends only on individual car parameters. Medium traffic may be viewed as a combination of light and heavy traffic. It consists of heterogeneous distribution of cars on the highway—freely-flowing regions together with traffic jams.

The use of a simulation along with the analysis allows students to visualize the more complex features of a model, while being grounded by fundamental principles. By sequentially following a series of example problems, the user can quickly appreciate the diverse phenomena associated with traffic flow. We also bring the attention of the user to some additional problems, for example the dynamics of traffic jams. This simulation tool has the flexibility to explore these and other interesting questions.

ACKNOWLEDGMENTS

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APPENDIX: DRAG FORCES

The functional dependence of the drag force on velocity is determined by the value of the dimensionless Reynolds number, defined as $R_e = \rho v_i l_i / \mu$, where ρ is the air density (~ 1.23 kg/m^3), μ is the air viscosity ($\sim 1.73 \times 10^{-5}$) Pa s, v_i is the velocity of the body, and l_i is the characteristic length. In our case l_i is the length of the car (~ 5 m). For most bodies, the drag is linearly related to velocity only if $R_e \lesssim 10$, the Stoke's flow regime. For values of $R_e \gtrsim 10^3$, the drag is very close to quadratic.[34]

For a typical range of velocities, we find the Reynolds number to be around 10^4 to 10^6, but, as shown below, the transient response varies little between the linear and quadratic form, given the same characteristic time. The use of the simpler linear form in the main body of the paper allows us to obtain simple analytical results (Sec. III A).

If the drag force increases quadratically with velocity, the equation of motion is

$$m_i \frac{d^2 x_i}{dt^2} + \alpha \left(\frac{dx_i}{dt} \right)^2 = F_i \, . \tag{A1}$$

For fixed F_i, the solution is

$$x_i(t) = C_2 + \frac{m_i \log[\cosh(\alpha t \sqrt{F_i} - C_1 m_i \sqrt{F_i}/m_i \sqrt{\alpha})]}{\alpha}. \tag{A2}$$

The velocity is

$$v_i(t) = \sqrt{\frac{F_i}{\alpha}} \tanh\left[\frac{-C_1 m_i \sqrt{F_i} + \alpha t \sqrt{F_i}}{m_i \sqrt{\alpha}} \right]. \tag{A3}$$

If $v(0) = 0$, then $C_1 = 0$, and

$$v_i(t) = v_\infty \tanh\left[\frac{t}{\tau} \right] \tag{A4a}$$

$$v_\infty = \sqrt{\frac{F_i}{\alpha}}, \tag{A4b}$$

where the characteristic time $\tau = m/\sqrt{F_I \alpha}$. The comparable transient result for linear drag is

$$v_i(t) = v_\infty [1 - \exp(t/\tau)]. \tag{A5}$$

A comparison of the linear and quadratic transient solutions shows that, for the same characteristic time τ, there is little difference between the two.

The power required to maintain a car at a given velocity v is ηv^2. For one of our cars traveling at 100 km/h, the power needed is about 130 hp, a rather high value. The reason for this is that we have assumed that the driver will maintain a constant force. It is more likely, of course, that the force applied by the driver decreases as the car approaches the desired velocity. As we show here, η represents an effective value that determines characteristic time, but not the power. One can simultaneously satisfy the required characteristic time and reasonable power usage at desired velocity by allowing the force to reduce as the velocity increases.

Imagine, for example, flooring the accelerator on entering a highway, and then backing off to a lower force. If we

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model this behavior as a linearly decreasing applied force, the governing equation of motion for a single car becomes

$$m_i \frac{d^2 x_i}{dt^2} + \eta_i \frac{dx_i}{dt} = F_i - (v_i - v_i^*) c. \tag{A6}$$

We rewrite Eq. (A6) as

$$m_i \frac{d^2 x_i}{dt^2} + \eta_i' \frac{dx_i}{dt} = F_i' , \tag{A7a}$$

$$\eta_i' = \eta_i + c, \tag{A7b}$$

$$F_i' = F_i + c v_i^* . \tag{A7c}$$

Therefore, even for a decreasing force, the dynamics can still be characterized by a simple, constant force and the viscosity equations of Sec. III A. For example, by choosing $\eta \approx 39$ kg/s, we can reduce the cruising power requirement at 100 kmph to 40 hp. By choosing $c \approx 85$, we can simultaneously achieve a characteristic time of 8 s. As far as the simulation is concerned, the cars move with effective viscosity of 125.

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[16] TrafficMaker, the accompanying manual and the source code are deposited at EPAPSDocument No. 15906. EPAPS is located at ⟨http://www.aip.org/pubservs/epaps.html⟩ or from ftp.aip.org in the directory /epaps/. See the EPAPS homepage for more information.

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[22] The accelerating force results from the engine exerting a torque on the wheels, which is matched by a force against the road acting on the radius of the tires. That is, the actual force propelling or braking the car is a real Newtonian force. The choice of its value comes from the driver in response to the desired velocity and the neighboring cars.

[23] The power required to maintain a car at a given velocity v is ηv^2. For one of our cars traveling at 100 km/h, the power needed is about 130 hp, which is rather high. The reason for this is that we have assumed that the driver will maintain a constant force. It is more likely, of course, that the force applied by the driver decreases as the car approaches the desired velocity. As shown in the Appendix, η represents an effective value that determines the characteristic time, but not the power. One can simultaneously satisfy the required characteristic time and reasonable power usage at desired velocity by allowing the force to reduce as the velocity increases.

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[25] To select a single car, select 1/2 Mile, 1 Lane, and Light Traffic from the choices at the top of the applet. Turn off Enable Auto-Add to prevent unwanted cars from being added. Pressing the Start button will create a single car driving on a one-lane highway. The Stop button clears the road, and Start will start a new car. Although each car has a different maximum desired speed, the desired speeds are distributed about 65 mph.

[26] Follow the same procedure as in Ref. 25 and select a single car on the road. Select Add a Broken Down Car. Clicking on the road will place a new broken-down car at the location of the click. The Remove button will clear all of the broken-down cars from the road, allowing traffic to flow again.

[27] Make sure Auto-Add is off. Select Add Normal Car and choose an initial speed for the new car. Once the first car has reached a steady state, add the second car by clicking on the desired starting location. Add the second car far away from the first car, so it will reach a steady state without interacting with the first car. Now two cars are moving independently of each other. If by chance both cars have the same desired speed, restart the simulation. Because of the periodic boundary conditions, the fast car will always become stuck behind the slow car.

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[29] As before, disable auto-add. After you have created a jam, click Remove to allow the cars to move again.

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[33] If the graphs are filled with stray points, use the Reset button to clear them. To produce the best curves, set the road length to 1 mile, start with one car, and then increase the number one at a time. Using the auto-add feature automates this process. The blue dot indicates the current state of the lane and the red dots past states. The darker the red dot, the longer the lane was at that particular state. Save the plots using screen capture software for later comparison. Reset the graphs and redo the curves, trying different waiting times for auto-add. Regardless of the auto-add rate, the curve traces out essentially the same path. Each graph can only plot approximately 1 hour’s worth of data. If the maximum number of points is exceeded, the new points will replace the older points in a first in, first out fashion.

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